SOLUTION In order to apply Equation 2, we first rewrite the function by multiplying and dividing by 7:

Note that $\sin 7x \neq 7 \sin x$.

$$\frac{\sin 7x}{4x} = \frac{7}{4}\left(\frac{\sin 7x}{7x}\right)$$

Notice that as $x \to 0$, we have $7x \to 0$, and so, by Equation 2 with $\theta = 7x$,

$$\lim_{x \to 0} \frac{\sin 7x}{7x} = \lim_{7x \to 0} \frac{\sin(7x)}{7x} = 1$$

Thus

$$\lim_{x \to 0} \frac{\sin 7x}{4x} = \lim_{x \to 0} \frac{7}{4}\left(\frac{\sin 7x}{7x}\right)$$

$$= \frac{7}{4} \lim_{x \to 0} \frac{\sin 7x}{7x} = \frac{7}{4} \cdot 1 = \frac{7}{4}$$

□

EXAMPLE 5 □ Calculate $\lim_{x \to 0} x \cot x$.

SOLUTION Here we divide numerator and denominator by x:

$$\lim_{x \to 0} x \cot x = \lim_{x \to 0} \frac{x \cos x}{\sin x}$$

$$= \lim_{x \to 0} \frac{\cos x}{\dfrac{\sin x}{x}} = \frac{\lim_{x \to 0} \cos x}{\lim_{x \to 0} \dfrac{\sin x}{x}}$$

$$= \frac{\cos 0}{1} \qquad \text{(by the continuity of cosine and Equation 2)}$$

$$= 1$$

□

3.4 Exercises

1–16 □ Differentiate.

Resources / Module 4 / Trigonometric Models / Derivatives of Trig Functions and Quiz

1. $f(x) = x - 3 \sin x$
2. $f(x) = x \sin x$
3. $y = \sin x + \cos x$
4. $y = \cos x - 2 \tan x$
5. $g(t) = t^3 \cos t$
6. $g(t) = 4 \sec t + \tan t$
7. $h(\theta) = \csc \theta + e^{\theta} \cot \theta$
8. $y = e^x \sin x$
9. $y = \dfrac{\tan x}{x}$
10. $y = \dfrac{\sin x}{1 + \cos x}$
11. $y = \dfrac{x}{\sin x + \cos x}$
12. $y = \dfrac{\tan x - 1}{\sec x}$
13. $y = \dfrac{\sin x}{x^2}$
14. $y = \tan \theta (\sin \theta + \cos \theta)$
15. $y = \csc x \cot x$
16. $y = x \sin x \cos x$

17. Prove that $\dfrac{d}{dx} (\csc x) = -\csc x \cot x$.
18. Prove that $\dfrac{d}{dx} (\sec x) = \sec x \tan x$.
19. Prove that $\dfrac{d}{dx} (\cot x) = -\csc^2 x$.
20. Prove, using the definition of derivative, that if $f(x) = \cos x$, then $f'(x) = -\sin x$.

21–24 □ Find an equation of the tangent line to the given curve at the specified point.

21. $y = \tan x$, $(\pi/4, 1)$
22. $y = 2 \sin x$, $(\pi/6, 1)$
23. $y = x + \cos x$, $(0, 1)$
24. $y = \dfrac{1}{\sin x + \cos x}$, $(0, 1)$

25. (a) Find an equation of the tangent line to the curve $y = x \cos x$ at the point $(\pi, -\pi)$.
(b) Illustrate part (a) by graphing the curve and the tangent line on the same screen.

26. (a) Find an equation of the tangent line to the curve $y = \sec x - 2\cos x$ at the point $(\pi/3, 1)$.
(b) Illustrate part (a) by graphing the curve and the tangent line on the same screen.

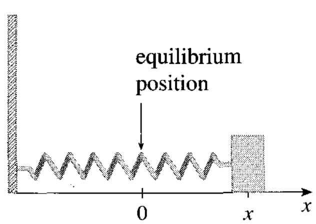
27. (a) If $f(x) = 2x + \cot x$, find $f'(x)$.
(b) Check to see that your answer to part (a) is reasonable by graphing both f and f' for $0 < x < \pi$.

28. (a) If $f(x) = \sqrt{x} \sin x$, find $f'(x)$.
(b) Check to see that your answer to part (a) is reasonable by graphing both f and f' for $0 \leqslant x \leqslant 2\pi$.

29. For what values of x does the graph of $f(x) = x + 2\sin x$ have a horizontal tangent?

30. Find the points on the curve $y = (\cos x)/(2 + \sin x)$ at which the tangent is horizontal.

31. A mass on a spring vibrates horizontally on a smooth level surface (see the figure). Its equation of motion is $x(t) = 8 \sin t$, where t is in seconds and x in centimeters.
(a) Find the velocity at time t.
(b) Find the position and velocity of the mass at time $t = 2\pi/3$. In what direction is it moving at that time?



32. An elastic band is hung on a hook and a mass is hung on the lower end of the band. When the mass is pulled downward and then released, it vibrates vertically. The equation of motion is $s = 2\cos t + 3 \sin t$, $t \geqslant 0$, where s is measured in centimeters and t in seconds. (We take the positive direction to be downward.)
(a) Find the velocity at time t.
(b) Graph the velocity and position functions.
(c) When does the mass pass through the equilibrium position for the first time?
(d) How far from its equilibrium position does the mass travel?
(e) When is the speed the greatest?

33. A ladder 10 ft long rests against a vertical wall. Let θ be the angle between the top of the ladder and the wall and let x be the distance from the bottom of the ladder to the wall. If the bottom of the ladder slides away from the wall, how fast does x change with respect to θ when $\theta = \pi/3$?

34. An object with weight W is dragged along a horizontal plane by a force acting along a rope attached to the object. If the rope makes an angle θ with the plane, then the magnitude of the force is

$$F = \frac{\mu W}{\mu \sin \theta + \cos \theta}$$

where μ is a constant called the *coefficient of friction*.
(a) Find the rate of change of F with respect to θ.
(b) When is this rate of change equal to 0?
(c) If $W = 50$ lb and $\mu = 0.6$, draw the graph of F as a function of θ and use it to locate the value of θ for which $dF/d\theta = 0$. Is the value consistent with your answer to part (b)?

35–44 □ Find the limit.

35. $\lim_{t \to 0} \dfrac{\sin 5t}{t}$

36. $\lim_{t \to 0} \dfrac{\sin 8t}{\sin 9t}$

37. $\lim_{\theta \to 0} \dfrac{\sin(\cos \theta)}{\sec \theta}$

38. $\lim_{\theta \to 0} \dfrac{\cos \theta - 1}{\sin \theta}$

39. $\lim_{\theta \to 0} \dfrac{\sin^2 \theta}{\theta}$

40. $\lim_{x \to 0} \dfrac{\tan x}{4x}$

41. $\lim_{x \to 0} \dfrac{\cot 2x}{\csc x}$

42. $\lim_{x \to \pi/4} \dfrac{\sin x - \cos x}{\cos 2x}$ (calculator)

43. $\lim_{\theta \to 0} \dfrac{\sin \theta}{\theta + \tan \theta}$

44. $\lim_{x \to 1} \dfrac{\sin(x - 1)}{x^2 + x - 2}$

45. Differentiate each trigonometric identity to obtain a new (or familiar) identity.
(a) $\tan x = \dfrac{\sin x}{\cos x}$
(b) $\sec x = \dfrac{1}{\cos x}$
(c) $\sin x + \cos x = \dfrac{1 + \cot x}{\csc x}$

46. A semicircle with diameter PQ sits on an isosceles triangle PQR to form a region shaped like an ice-cream cone, as shown